Chapter 7: Applications of Residue Theory

We will now apply the theory of residues to compute several types of improper integrals from real analysis.

Additionally, we will prove:

(1) Argument Principle - The winding number of the image of a curve under certain analytic functions depends only on the number of zeros and poles of that function.

(2) Rouche's Theorem - A useful criterion for locating the zeros of an analytic function.

Background on Improper Integrals

Definition Suppose $f(x)$ is a real-valued function of a real variable.

(a) If $f(x)$ is continuous on $[0, \infty)$, then the improper integral of f over that interval is defined to be

$$\int_0^\infty f(x)\,dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_0^R f(x)\,dx.$$

If the limit exists, the integral is said to converge.

(b) If $f(x)$ is continuous on $\mathbb{R}$, then the improper integral of f over $\mathbb{R}$ is defined via

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} f(x)\,dx$$

The integral converges if both limits exist.

(c) The Cauchy Principal Value of the improper integral in (b) is the value of the limit

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx .$$

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$\boxed{\text{Lemma}}$ If $\int_{-\infty}^{\infty} f(x)\,dx$ converges, then the Cauchy principal value exists and $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$.

Proof. Just notice

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{R \to \infty} \int_{0}^{R} f(x)\,dx + \lim_{R \to \infty} \int_{-R}^{0} f(x)\,dx \\
&= \lim_{R \to \infty} \left(\int_{0}^{R} f(x)\,dx + \int_{-R}^{0} f(x)\,dx \right) \\
&= \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx .
\end{aligned}$$

∎

The converse is false - even if the Cauchy Principal value exists, the integral may diverge.

$\boxed{\text{Lemma}}$ Suppose $f(x)$ is continuous on $\mathbb{R}$ and even. If $P.V. \int_{-\infty}^{\infty} f(x)\,dx$ exists, then the improper integral exists and

$$\int_{-\infty}^{\infty} f(x)\,dx = P.V. \int_{-\infty}^{\infty} f(x)\,dx.$$

Moreover,
$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\; P.V. \int_{-\infty}^{\infty} f(x)\,dx.$$

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,dx \\
&= \lim_{R_1\to\infty} \frac{1}{2} \int_{-R_1}^{R_1} f(x)\,dx + \lim_{R_2\to\infty} \frac{1}{2} \int_{-R_2}^{R_2} f(x)\,dx \\
&= \frac{1}{2}\, P.V. \int_{-\infty}^{\infty} f(x)\,dx + \frac{1}{2}\, P.V. \int_{-\infty}^{\infty} f(x)\,dx \\
&= P.V. \int_{-\infty}^{\infty} f(x)\,dx.
\end{aligned}$$

Along the way, we also proved
$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\, P.V. \int_{-\infty}^{\infty} f(x)\,dx.$$

$\blacksquare$

Improper Integrals of Rational Functions

Assumptions:

(1) $f(x) = \frac{p(x)}{q(x)}$ is a rational function with real coefficients and such that $p(x)$ and $q(x)$ have no factors in common.

(2) $q(x)$ has no real zeros and atleast one zero with positive imaginary part.

(3) $f(x)$ is an even function.

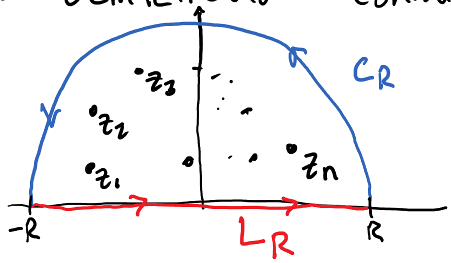
We describe a method to compute the integrals:

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx \qquad \text{and} \qquad \int_{0}^{\infty} \frac{p(x)}{q(x)}\,dx .$$

Step 1: Identify the singularities of f that lie above the real axis. By assumption, there is at least one. Label them

$$z_1, \dots, z_n.$$

Step 2: Define a semicircular contour C as follows:



$$C = C_R + L_R$$

C_R: the semicircle parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$

L_R: the line segment joining $-R$ to R.

Choose $R>0$ such that $R > \max_{i=1}^{n} |z_i|$.

Step 3: Apply the residue theorem:

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = \int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z).$$

Parametrize L_R via $z(x) = x$, $-R \leq x \leq R$. Then

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx$$

Hence,

$$P.V \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z) - \lim_{R\to\infty} \int_{C_R} f(z)\,dz.$$

Since $f(x)$ is even $\int_{-\infty}^{\infty} f(x)\,dx = P.V. \int_{-\infty}^{\infty} f(x)\,dx$.

Step 4: Prove that $\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\,dz = 0$. This can always be proved if, for instance, $\deg p(z) + 2 \leq \deg q(z)$.

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$\boxed{\text{Example}}$ Compute $\displaystyle\int_0^\infty \frac{1}{x^4+1}\,dx$.

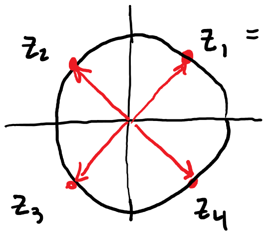
The singularities of $f(z)=\frac{1}{z^4+1}$ are the solutions of $z^4=-1$.

$$(-1)^{1/4} = e^{\frac{1}{4}\log(-1)} = e^{\frac{1}{4}(\ln|-1| + i\arg -1)}$$

$$= e^{\frac{1}{4}i(\pi+2k\pi)}, \quad k=0,1,2,3.$$

$$= e^{\frac{i\pi}{4}}\cdot e^{\frac{ik\pi}{2}}$$

$-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} = z_2$ $\qquad$ $z_1 = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$



Integrate $f(z)$ over the semicircular contour with $R>1$. By the residue theorem, we get

$$\int_{-R}^{R}\frac{1}{x^4+1} = 2\pi i\left(\operatorname*{Res}_{z=z_1}\frac{1}{z^4+1} + \operatorname*{Res}_{z=z_2}\frac{1}{z^4+1}\right) - \int_{C_R}\frac{1}{z^4+1}\,dz.$$

Let $p(z)=1$ and $q(z)=z^4+1$. Then p,q are both analytic at each singularity z_k, $p(z_k)=1\neq 0$, $q(z_k)=0$, and $q'(z_k)=4z_k^3\neq 0$. Hence, each z_k is a simple poles with residue given by

$$\operatorname*{Res}_{z=z_k}\frac{1}{z^4+1} = \frac{p(z_k)}{q'(z_k)} = \frac{1}{4z_k^3} = \frac{z_k}{4z_k^4} = -\frac{z_k}{4}.$$

Next,

$$\left|\int_{C_R}\frac{1}{z^4+1}\,dz\right| \le \pi R\cdot \max_{|z|=R}\frac{1}{|z^4+1|}$$

$$\le \frac{\pi R}{R^4-1} \xrightarrow{R\to\infty} 0.$$

$$|z^4+1| \ge \big||z|^4-1\big| = |R^4-1| = R^4-1$$

Hence,
$$\int_0^{\infty} \frac{1}{x^4+1}\,dx = \frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$
$$= \pi i \left(\operatorname*{Res}_{z=z_1} \frac{1}{z^4+1} + \operatorname*{Res}_{z=z_2} \frac{1}{z^4+1} \right)$$
$$= -\frac{\pi i}{4}\left(z_1 + z_2 \right)$$
$$= -\frac{\pi i}{4}\left(\cancel{\frac{\sqrt{2}}{2}} + i\frac{\sqrt{2}}{2} + \cancel{-\frac{\sqrt{2}}{2}} + i\frac{\sqrt{2}}{2} \right) = \frac{\pi\sqrt{2}}{4}. \quad //$$

Improper Integrals from Fourier Analysis

Assumptions :

(1) $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function with real coefficients and such that $p(x)$ and $q(x)$ have no factors in common.

(2) $q(x)$ has no real zeros and atleast one zero with positive imaginary part.

(3) $a > 0$ and $f(z)\sin az$ $\left(\text{or } f(z)\cos az\right)$ is an even function.

The same method, with a slight modification, can be used to compute the integral
$$\int_{-\infty}^{\infty} f(x)\sin ax\,dx \qquad \left(\text{or } \int_{-\infty}^{\infty} f(x)\cos ax\,dx \right).$$

We use Euler's formula to write

$$\int_{-R}^{R} f(x)\cos ax\,dx + i\int_{-R}^{R} f(x)\sin ax\,dx = \int_{-R}^{R} f(x)e^{iax}\,dx$$

We will simply compute the RHS, take the real or imaginary part, and then take the limit.

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$\boxed{\text{Example}}$ Compute $\displaystyle\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx$. We will integrate $f(z)e^{2iz}$ where $f(z)=\frac{1}{(x^2+4)^2}$ over the semicircular contour C. Clearly, $f(z)$ has a single singularity at $z=2i$ that lies above the real axis. Assuming $R>2$, $f(z)e^{2iz}$ is analytic inside and on C except at a single point. By the residue theorem

$$\int_{-R}^{R} f(x)e^{2ix}\,dx = 2\pi i \operatorname*{Res}_{z=2i} f(z)e^{2iz} - \int_{C_R} f(z)e^{2iz}\,dz.$$

To compute the residue, define $\phi(z) = \dfrac{e^{2iz}}{(z+2i)^2}$ so that

$f(z)\,e^{2iz} = \dfrac{\phi(z)}{(z-2i)^2}$. Moreover, $\phi(z)$ is nonzero and analytic at $z=2i$. Hence $z=2i$ is a pole of order $m=2$ and

$$\operatorname*{Res}_{z=2i} f(z)\,e^{2iz} = \frac{\phi^{(2-1)}(2i)}{(2-1)!} = \phi'(2i).$$

We have

$$\phi'(2i) = \left.\frac{2ie^{2iz}(z+2i)^2 - 2(z+2i)e^{2iz}}{(z+2i)^4}\right|_{2i} = \frac{2ie^{-4}(4i)^2-2(4i)e^{-4}}{(4i)^4}$$

$$= \frac{2(4i)e^{-4}(4i^2-1)}{(4i)^4} = \frac{2e^{-4}(-5)}{(4i)^3} = \frac{5}{32i}e^{-4}.$$

Then we have

$$\left|\int_{C_R} f(z)\cos 2z\right| \le \left|\int_{C_R} \frac{1}{(z^2+4)^2} e^{2iz}\,dz\right| \le \pi R \max_{|z|=R} \frac{|e^{2iz}|}{|z^2+4|^2}$$

$$\le \frac{\pi R}{R^2-4} \xrightarrow{R\to\infty} 0.$$

$\operatorname{Im} z > 0$

$$|e^{2iz}| = |e^{2ix}||e^{-2y}| = |e^{-2y}| \le 1$$

$$|z^2+4| \ge |z|^2-4 = R^2-4$$

We have

$$\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{1}{2}\,\mathrm{P.V}\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+4)^2}$$

$$= \frac{1}{2}\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty} \operatorname{Re}\int_{-R}^{R} f(x)e^{2ix}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty}\left(\frac{10\pi i}{32i}e^{-4} - \int_{C_R} f(z)e^{2iz}\,dz\right)$$

$$= \frac{5\pi}{32}e^{-4}.$$

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The preceding method works if

$$\deg p(x) + 2 \le \deg q(x).$$

If not, the triangle inequality for contour integrals may not be enough to prove

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{aiz}\,dz = 0.$$

In this case, you may be able to use Jordans lemma instead:

Lemma (Jordan) Assume

(1) f is analytic at all points in the upper half plane ($\operatorname{Im} z > 0$) that are exterior to some circle $|z| = R_0$.

(2) C_R is the semicircle ($z(t) = Re^{it}$ $0 \le t \le \pi$) with $R > R_0$.

(3) There exists $M_R > 0$ such that $|f(z)| \le M_R$ for all $z \in C_R$ and $\lim_{R \to \infty} M_R = 0$.

Then for any $a > 0$,

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{aiz} \, dz = 0.$$

Proof. Assume this w/out proof. See the book. $\square$

Example Compute $\displaystyle\int_0^\infty \frac{x \sin 2x}{x^2+3} \, dx$.

Let $f(z) = \dfrac{z}{z^2+3} = \dfrac{z}{(z-\sqrt{3}i)(z+\sqrt{3}i)}$. We integrate $f(z)e^{2iz}$ over the semicircular contour C. The only singularity lying above the real axis is $z = \sqrt{3}i$.

Compute the residue: write $p(z) = ze^{2iz}$ and $q(z) = z^2+3$. Both are analytic at $z = \sqrt{3}i$, $p(\sqrt{3}i) \ne 0$, $q(\sqrt{3}i) = 0$, $q'(\sqrt{3}i) = 2\sqrt{3}i \ne 0$. So $z = \sqrt{3}i$ is a simple pole with residue

$$\operatorname*{Res}_{z=\sqrt{3}i} f(z)e^{2iz} = \frac{p(\sqrt{3}i)}{q'(\sqrt{3}i)} = \frac{1}{2} e^{-2\sqrt{3}}.$$

By the residue theorem,

$$\int_{-R}^{R} \frac{x \sin 2x}{x^2+3}\,dx = \mathrm{Im} \int_{-R}^{R} f(x) e^{2ix}\,dx$$
$$= \mathrm{Im}\left(\pi i e^{-2\sqrt{3}} - \int_{C_R} f(z) e^{2iz}\,dz \right)$$
$$= \pi e^{-2\sqrt{3}} - \mathrm{Im} \int_{C_R} f(z) e^{2iz}\,dz.$$

So, we just need to show that

$$\lim_{R\to\infty} \mathrm{Im} \int_{C_R} \frac{z e^{2iz}}{z^2+3}\,dz = 0.$$

For any $|z| = R$, we have

$$\left| \frac{z}{z^2+3} \right| = \frac{|z|}{|z^2+3|} \le \frac{|z|}{|z|^2-3} = \frac{R}{R^2-3} =: M_R.$$

Since $\lim_{R\to\infty} M_R$, by Jordans Lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{z e^{2iz}}{z^2+3}\,dz = 0$$

which implies

$$\lim_{R\to\infty} \mathrm{Im} \int_{C_R} \frac{z e^{2iz}}{z^2+3}\,dz = 0.$$

Hence,

$$\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx = \frac{1}{2}\, \mathrm{P.V.} \int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx$$
$$= \frac{1}{2} \pi e^{-2\sqrt{3}}$$

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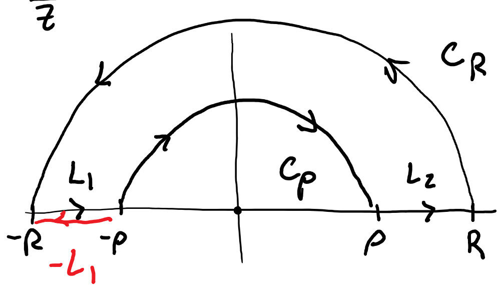
Indented Path

An indented path can sometimes be used to avoid an isolated singularity or branch point that lies on the real axis. We illustrate this method through an example.

$\boxed{\text{Example (Dirichlet's Integral)}}$ Compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

We integrate $f(z) = \dfrac{e^{iz}}{z}$ over the contour



$C = C_R + L_1 + C_\rho + L_2$

By Cauchy-Goursat

$$0 = \int_C \frac{e^{iz}}{z}\,dz = \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{C_\rho} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz.$$

Parametrize $-L_1, L_2$ as follows

$$-L_1: \quad z(t) = -t, \quad \rho \le t \le R$$
$$L_2: \quad z(t) = t, \quad \rho \le t \le R.$$

Then

$$\int_{L_2} \frac{e^{iz}}{z}\,dz + \int_{L_1} \frac{e^{iz}}{z}\,dz = \int_{L_2} \frac{e^{iz}}{z}\,dz - \int_{-L_1} \frac{e^{iz}}{z}\,dz$$
$$= \int_\rho^R \frac{e^{it}}{t}\,dt - \int_\rho^R \frac{e^{-it}}{t}\,dt$$

$$= \int_\rho^R \frac{e^{it} - e^{-it}}{t}\, dt$$

$$= 2i \int_\rho^R \frac{\sin t}{t}\, dt.$$

Hence,

$$\int_\rho^R \frac{\sin x}{x}\, dx = -\frac{1}{2i}\left(\int_{C_R} \frac{e^{iz}}{z}\, dz + \int_{C_\rho} \frac{e^{iz}}{z}\, dz \right)$$

Thus,

$$\int_0^\infty \frac{\sin x}{x}\, dx = -\frac{1}{2i}\left(\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz + \lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}}{z}\, dz \right).$$

By Jordans Lemma, with $M_R := \frac{1}{R}$, $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz = 0.$

To compute $\lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}}{z}\, dz$, consider the Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} \sum_{n=0}^\infty \frac{(iz)^n}{n!} = \sum_{n=0}^\infty \frac{i^n z^{n-1}}{n!} = \frac{1}{z} + \sum_{n=1}^\infty \frac{i^n z^{n-1}}{n!}.$$

$$= \frac{1}{z} + g(z).$$

Since $g(z)$ is a Taylor series about zero, it is analytic in a neighborhood of zero. Hence, there is a closed disk $|z| \le \varepsilon$ on which $g(z)$ is continuous. Hence, $g(z)$ is bounded near zero, say

$$|g(z)| \le M \quad \text{for all } |z| \le \varepsilon.$$

Hence, if $\rho < \varepsilon$,

$$\left|\int_{C_\rho} g(z)\,dz\right| \overset{T.I.}{\leq} \pi \cdot \rho \max_{z\in C_\rho} |g(z)|$$

$$\leq \pi \cdot \rho \cdot M \xrightarrow{\rho\to 0} 0.$$

Thus,
$$\lim_{\rho\to 0}\int_{C_\rho}\frac{e^{iz}}{z}\,dz = \lim_{\rho\to 0}\int_{C_\rho}\frac{1}{z}\,dz + \lim_{\rho\to 0}\int_{C_\rho} g(z)\,dz = -\pi i + 0$$

Finally,
$$\int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}(-\pi i) = \frac{\pi}{2}.$$

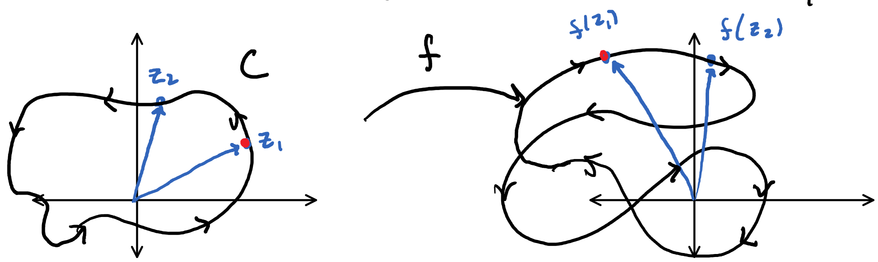
Argument Principle

A function is meromorphic on a domain D if it is analytic on D except for poles.

Let C be a simple closed positively oriented contour and denote by D its interior. Suppose f is meromorphic on D and analytic and nonzero on C. The image

$$f(C) = \{ f(z) : z \in C \}$$

of C under f is also a closed contour, but not necessarily simple.



Since f is nonzero on C, it follows that f(C) does not cross the origin. Consequently, the winding number of f(C) about $z=0$ is defined:

$$n(f(C),0) = \frac{1}{2\pi i}\int_{f(C)} \frac{1}{z}\,dz.$$

In PSet 5, we saw that the winding number is interpreted geometrically as the number of times the contour winds around a point. We can write

$$n(f(C),0) = \frac{1}{2\pi}\Delta_C \operatorname{Arg} f(z)$$

where $\Delta_C \operatorname{Arg} f(z)$ is the change in argument of $f(z)$ as C is traversed once in the positive direction.

The argument principle shows that $n(f(C),0)$ depends only on the number of poles and zeros, counting orders, that lie interior in D.

Theorem (Argument Principle) Let C be a simple closed positively oriented contour and D its interior. Suppose that

(1) f is meromorphic on D;

(2) f is analytic and nonzero on C and not identically zero on D;

(3) Counting orders, Z is the number of zeros of f in D, and P is the number of poles of f in D.

Then
$$n(f(C), 0) = Z - P.$$

Proof. First of all, the number of poles of f in D is finite. Label the poles:
$$p_1, \cdots, p_K \quad \text{with orders} \quad m_1, \dots, m_K.$$

Also, the number of zeros is finite since f is not identically zero on D. Label the zeros:
$$z_1, \cdots, z_\ell \quad \text{with orders} \quad n_1, \cdots, n_\ell.$$

Note that: $Z = \sum_{j=1}^{\ell} n_j$ and $P = \sum_{j=1}^{K} m_j$. Now, consider the function $\frac{f'(z)}{f(z)}$. Note that $\frac{f'(z)}{f(z)}$ is analytic on C and on D except possibly at the zeros or poles of f.

Since f has a zero of order n_j at z_j, we can write
$$f(z) = (z - z_j)^{n_j} g(z)$$
for some $g(z)$ that is analytic and nonzero at z_j.

Then
$$\frac{f'(z)}{f(z)} = \frac{n_j (z-z_j)^{n_j - 1} g(z) + (z - z_j)^{n_j} g'(z)}{(z - z_j)^{n_j} g(z)}$$
$$= \frac{n_j}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Since $\frac{g'(z)}{g(z)}$ is analytic at z_j, this shows that $\frac{f'(z)}{f(z)}$ has a simple pole at z_j with

$$\operatorname*{Res}_{z=z_j} \frac{f'(z)}{f(z)} = n_j .$$

Similarly, at each pole p_j w/ order m_j, we can write

$$f(z) = \frac{\phi(z)}{(z-p_j)^{m_j}} = (z-p_j)^{-m_j} \phi(z)$$

where $\phi(z)$ is analytic and nonzero at p_j. Then

$$\frac{f'(z)}{f(z)} = \frac{-m_j (z-p_j)^{-m_j-1}\phi(z) + (z-p_j)^{-m_j}\phi'(z)}{(z-p_j)^{-m_j}\phi(z)}$$

$$= \frac{-m_j}{z-p_j} + \frac{\phi'(z)}{\phi(z)} .$$

Hence, p_j is a simple pole of $\frac{f'(z)}{f(z)}$ w/ residue

$$\operatorname*{Res}_{z=p_j} \frac{f'(z)}{f(z)} = -m_j .$$

Then we obtain

$$\begin{aligned} n(f(C),0) &= \frac{1}{2\pi i}\int_{f(C)} \frac{1}{z}\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz \\ &= \sum_{j=1}^{l} \operatorname*{Res}_{z=z_j}\frac{f'(z)}{f(z)} + \sum_{j=1}^{K} \operatorname*{Res}_{z=p_j}\frac{f'(z)}{f(z)} \\ &= \sum_{j=1}^{l} n_j - \sum_{j=1}^{K} m_j \\ &= Z - P. \end{aligned}$$

$\blacksquare$

$\boxed{\text{Example}}$ Let C' be a contour parameterized via

$$w(t) = \frac{(2e^{it}-1)^7}{e^{3it}}, \quad 0 \le t \le 2\pi.$$

Compute $n(C', 0)$.

Consider $f(z) = \frac{(2z-1)^7}{z^3}$. Then $f(C) = C'$ where

C is the unit circle $(z(t) = e^{it},\ 0 \le t \le 2\pi)$. By the argument principle,

$$\begin{aligned} n(C', 0) &= n(f(C), 0) \\ &= Z - P. \end{aligned}$$

To compute Z: write $g(z) = \frac{2^7}{z^3}$. Then $f(z) = (z - \frac{1}{2})^7 g(z)$

and $g(z)$ is analytic and nonzero at $z = \frac{1}{2}$. So z is a zero of order 7 of f, and it lies interior to C. So $Z = 7$.

To compute P: define $\phi(z) = (2z-1)^7$. Then $f(z) = \frac{\phi(z)}{z^3}$

and ϕ is analytic and nonzero at $z=0$. So $z=0$ is a pole of order 3 and $z=0$ lies interior to C. So $P = 3$.

Hence,
$$\begin{aligned} n(C', 0) &= 7 - 3 \\ &= 4. \end{aligned}$$

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Rouche's Theorem

Theorem (Rouche) Let C be a simple closed contour and suppose that

(1) $f(z)$ and $g(z)$ are analytic on and interior to C;
(2) $|f(z)| > |g(z)|$ for all $z \in C$.

Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting orders, interior to C.

Proof. By (2), $|f(z)| > |g(z)| \geq 0$ for all $z \in C$, so f has no zeros on C. Moreover, for all $z \in C$,

$$|f(z)+g(z)| \geq \big|\, |f(z)| - |g(z)| \,\big| > 0 .$$

Hence, also $f+g$ has no zeros on C. Denote by Z_f and Z_{f+g} the number of zeros of f and $f+g$, counting multiplicities, that lie interior to C. By the argument principle

$$\begin{aligned} Z_{f+g} &= n\left((f+g)(C), 0\right) \\ &= \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(f(z)+g(z)\right) \\ &= \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(f(z)\left(1+\frac{g(z)}{f(z)}\right)\right) \\ &= \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(f(z)\right) + \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right) \\ &= Z_f + n\left(F(C), 0\right) \end{aligned}$$

where $F(z) = 1 + \frac{g(z)}{f(z)}$. Now, let $z \in C$. Then

$$|F(z)-1| = \frac{|g(z)|}{|f(z)|} < 1.$$

This proves that $F(C)$ is contained in the open disk $D_1(1)$. Since $0 \notin D_1(1)$, we conclude that $n(F(C),0)=0$. This proves

$$Z_f = Z_{f+g}.$$

$\blacksquare$

$\boxed{\text{Example}}$ Determine the number of zeros, counting orders, of

$$p(z) = z^7 - 4z^3 + z - 1$$

that lie interior to the unit circle $|z|=1$.

The strategy: Choose $f(z)$ and $g(z)$ as in Rouche's theorem, such that

$$p(z) = f(z) + g(z).$$

Moreover, the number of zeros of f interior to C should be easy to compute.

For this problem, define $f(z) = z^7 - 4z^3$ and $g(z) = z-1$. Note that $f(z)+g(z) = p(z)$ and $f(z)$ has three zeros interior to C. Let $z \in C$ so that $|z|=1$. Then

$$\begin{aligned} |f(z)| = |z^7 - 4z^3| &= |z|^3 |z^4 - 4| \\ &\geq 1 \cdot \big| |z|^4 - 4 \big| = |1-4| = 3. \end{aligned}$$

Also,

$$|g(z)| = |z-1| \leq |z| + 1 = 2.$$

Hence, $|f(z)| \geq 3 > 2 \geq |g(z)|$. By Rouche's theorem, $p(z)$ has three zeros interior to C.

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$\boxed{\text{Example}}$ Determine the number of zeros, counting orders, of

$$p(z) = 2z^5 - 6z^2 + z + 1$$

that lie in the annulus $1 \le |z| < 2$.

First, compute the number of zeros interior to the circle $C_2(0)$. Take $f(z) = 2z^5 + 1$ and $g(z) = 1 - 6z^2$. If $|z| = 2$, then

$$|f(z)| = |2z^5 + 1| \ge \left|2|z|^5 - 1\right| = |2 \cdot 2^5 - 1| = 63.$$

Also,

$$|g(z)| = |1 - 6z^2| \le 1 + 6|z|^2 = 25.$$

Hence, $|f(z)| \ge 63 > 25 \ge |g(z)|$ so by Rouché $p(z)$ has five zeros interior to $C_2(0)$ since $f(z)$ does.

Next, consider the unit circle C. Let $f(z) = -6z^2$ and $g(z) = 2z^5 + z + 1$. Then if $|z| = 1$,

$$|f(z)| = 6|z|^2 = 6,$$

while

$$|g(z)| \le 2|z|^5 + |z| + 1 = 4.$$

So $|f(z)| = 6 > 4 \ge |g(z)|$. By Rouché, $p(z)$ has 2 zeros interior to C since $f(z)$ does. Hence, $p(z)$ has $5 - 2 = 3$ zeros in the annulus. //